1. (30 points) The following problems are not related.

   (a) (10 points) Find the derivative of $g(x) = \sin\left(\frac{x^2 + x}{3x - 1}\right)$. Do not simplify your answer.

   (b) (14 points) Let $f(x) = \sqrt{4 - x}$.

      i. State the limit definition of the derivative for a function $f(x)$.

      ii. Find $f'(x)$ by using the definition of the derivative. *You must use the limit definition to receive any credit.*

   (c) (6 points) If $f'(x) = \lim_{h \to 0} \frac{\sin(x + h) - \sin(x)}{h}$, find $f'(\ /3)$.

Solution:

(a)

$$g'(x) = \cos\left(\frac{x^2 + x}{3x + 1}\right) \cdot \left(\frac{(3x - 1)(2x + 1) - (x^2 + x)(3)}{(3x - 1)^2}\right).$$

(b) i. The derivative of a function $f(x)$ is defined to be

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

so long as the limit exists.

ii. Using the limit definition:

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{\sqrt{4 - (x + h)} - \sqrt{4 - x}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{4 - (x + h)} - \sqrt{4 - x}}{h} \cdot \frac{\sqrt{4 - (x + h)} - \sqrt{4 - x}}{\sqrt{4 - (x + h)} - \sqrt{4 - x}} \\
&= \lim_{h \to 0} \frac{4 - (x + h) - 4 - x}{h(\sqrt{4 - (x + h)} - \sqrt{4 - x})} \\
&= \lim_{h \to 0} \frac{-h}{h(\sqrt{4 - (x + h)} - \sqrt{4 - x})} \\
&= \lim_{h \to 0} \frac{-1}{\sqrt{4 - (x + h)} - \sqrt{4 - x}} \\
&= -\frac{1}{2\sqrt{4 - x}}.
\end{aligned}$$

(c) The right-hand side is the limit definition of the derivative for $\sin(x)$, so $f(x) = \sin(x)$. We know that $f'(x) = \cos(x)$, so $f'(\ /3) = \cos(\ /3) = 1/2$.

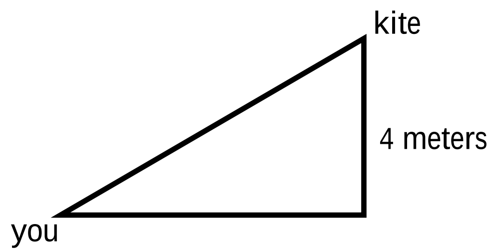
2. (20 points) The following problems are not related.

   (a) (8 points) The side length h of a square is measured as 3 cm, with a maximum error of 0.1 cm. Use differentials to estimate:

i. the maximum error for the area of the square;

ii. the relative error for the area of the square.

(b) (12 points) You are flying a kite which has a constant height of 4 meters above the ground. The wind is carrying the kite horizontally away from you, and you have to let out string at a rate of 2 meters/minute. What is the horizontal speed of the kite when you have let out 5 meters of string?



Solution:

(a) i. The area of a square is given by $A(h) = h^2$, so we have that

$$dA = 2h\,dh = (2)(3)(0.1) = 0.6.$$

Hence, the maximum error for the area of the square is 0.6 cm^2 in this situation.

ii. For a side length measurement of 3 cm, the area is 9 cm^2, so the relative error for the area is

$$\frac{dA}{A} = \frac{0.6}{9} = \frac{3}{5} \cdot \frac{1}{9} = \frac{1}{15} \approx 6.\overline{66}\%.$$

(b) Letting z be the hypotenuse (the distance from you to the kite), and x the horizontal distance, we know that

$$z^2 = x^2 + 4^2 \implies 2z\frac{dz}{dt} = 2x\frac{dx}{dt} \implies \frac{dx}{dt} = \frac{z}{x} \cdot \frac{dz}{dt}.$$

In order to get a value for $\frac{dx}{dt}$, we first need to get the value of x when $z = 5$:

$$5^2 = x^2 + 4^2 \implies 25 = x^2 + 16 \implies 9 = x^2 \implies x = 3.$$

Hence, when $z = 5$, we have that

$$\frac{dx}{dt} = \frac{z}{x} \cdot \frac{dz}{dt} = \frac{5}{3} \cdot (2) = \frac{10}{3} \text{ meters/min}.$$

3. (16 points) Consider the function $s(x) = -x^3 + 3x + 2$.

(a) Find the critical numbers of $s(x)$.

(b) Use the first derivative test to determine the points where $s(x)$ has a local maximum or local minimum. *Give your answer as ordered pairs* (x, y).

(c) Find the absolute maximum and minimum values for the function $s(x)$ on the interval $[0, 2]$.

Solution:

(a) To find the critical numbers, first take the derivative

$$s'(x) = -3x^2 + 3.$$

Since the domain of $s'(x)$ is $(-\infty, \infty)$, the only critical numbers are solutions to the equation

$$0 = -3x^2 + 3,$$

and hence

$$0 = -3x^2 + 3 \implies 3x^2 = 3 \implies x^2 = 1 \implies x = \pm 1.$$

So $x = \pm 1$ are the only critical numbers. The function values at the critical numbers are $s(-1) = 0$, $s(1) = 4$

(b) In order to determine whether each critical number $x = \pm 1$ is a local maximum, minimum, or neither, we apply the first derivative test to the intervals $(-\infty, -1)$, $(-$

which implies that $x = 2$. Hence, we have to find the tangent line at the point $(2; 0)$. Plugging these values into the formula for $\frac{dy}{dx}$, we find that

$$\left.\frac{dy}{dx}\right|_{(x;y)=(2;0)} = -\frac{1=2 - 0}{0 + 1} = -\frac{1}{2}:$$

Then an equation for the tangent line to the curve at $(2; 0)$ is given by

$$y = -\frac{1}{2}(x - 2):$$

5. (16 points) Consider the function $f(x) = \frac{1}{x}$ on the interval $[2; 4]$.

   (a) (8 points) State the Mean Value Theorem and verify that $f(x)$ satisfies the hypotheses on the given interval.

   (b) (8 points) Find all numbers c that satisfy the conclusion of the Mean Value Theorem for $f(x)$ on the interval $[2; 4]$.

Solution:

(a) If $f(x)$ is continuous on $[a; b]$ and differentiable on $(a; b)$, then there is a c in the interval $(a; b)$ such that

$$f^0(c) = \frac{f(b) - f(a)}{b - a}:$$

The only value where $f(x)$ is discontinuous is $x = 0$, so $f(x)$ is continuous on $[2; 4]$. The function $f(x)$ is differentiable on $[2; 4]$, since $f^0(x) = -\frac{1}{x^2}$, which is undefined only at $x = 0$.;r0(Mean)-2(Mean50(oh)-25ean)-2(M